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UNIVERSITY OF GLOBAL VILLAGE (UGV), BARISHAL.

THE FIRST SKILL BASED HI-TECH UNIVERSITY IN BANGLADESH

Discrete Mathematics

Lecture Prepared By

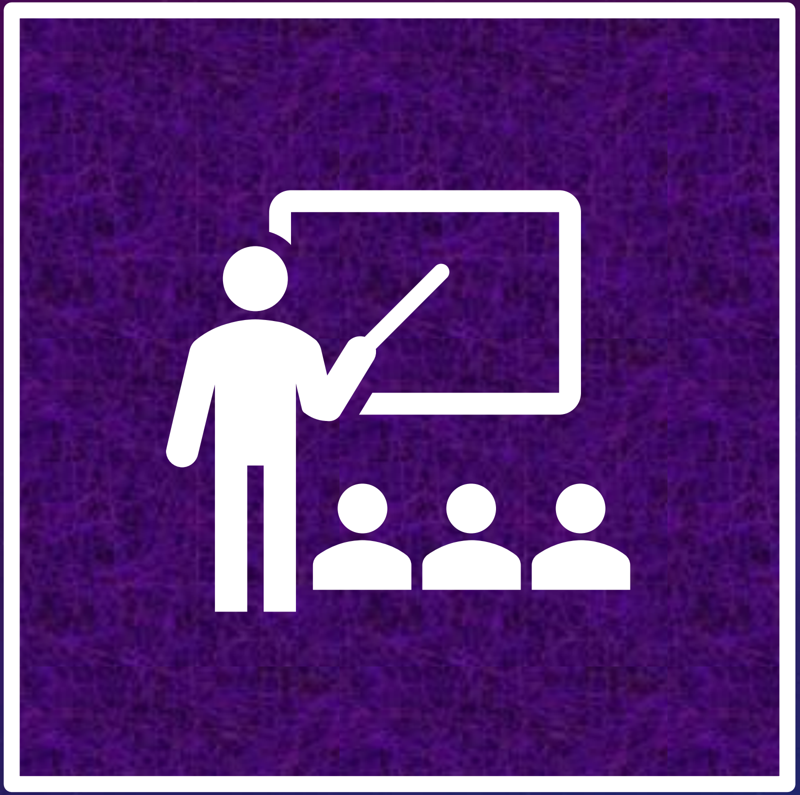
Md. Shakibul Ajam

Lecturer

Department of Mathematics

University of Global Village (UGV), Barishal

COURSE DETAILS



Curriculum	: Outcome Based Education
Course Name	: Discrete Mathematics
Course Code	: CSE 0541-1203
Teacher’s Name	: Md. Shakibul Ajam
Course Credit	: 03
CIE Marks	: 90 (60%)
SEE Marks	: 60 (40%)
SEE Duration	: 03 Hours

RATIONALE OF THIS COURSE

This course is designed to introduce the students to the basic concepts of discrete mathematics. The focus is to illustrate engineering applications of these principles. The learning approach is how the students can deal the engineering problems related to this course.

Students will achieve comprehension of the fundamental knowledge from this course and they will be able to apply it in the branch of engineering.

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COURSE LEARNING OUTCOME (CLO)

After successful completion of the course students will be able to -

1) Acquire proficiency in fundamental concepts of discrete mathematics, including sets, relations, functions, and logic.
2) Develop advanced analytical and problem-solving skills essential for tackling complex computational challenges.
3) Demonstrate mastery in logical reasoning and the formulation of rigorous mathematical proofs.
4) Apply theoretical knowledge to effectively solve practical problems.

CONTENT OF THE COURSE

SL.	Content of Courses	Hrs	CLO's
01	**Set Theory:** Sets and elements, Subsets, Universal Sets, Empty Set, Disjoint Sets, Venn Diagrams, Set Operations, Algebra of Sets, Finite Sets, Counting Principle, The Inclusion-Exclusion Principle, Classes of Sets, Power Sets, Partitions, Mathematical Inductions. **Relations:** Product Sets, Relations, Composition of Relations, Types of Relations, Equivalence Relations, Partial Ordering Relations	12	01, 03
02	**Functions and Algorithms:** Functions, One-To-One, Onto and Invertible Functions, Mathematical Functions, Exponential and Logarithmic Functions	4	01, 03
03	**Logic and Propositional Calculus:** Propositions and Compound Statements, Basic Logical Operations, Proposition and Truth Tables, Tautologies and Contradictions, Algebra of Propositions, Conditionals and Biconditional Statements. **Boolean Algebra:** Logic Gates.	8	02, 03
04	**Techniques of Counting:** Basic Counting Principles, Permutations, Combinations, The Pigeonhole Principle.	4	03, 04
05	**Graph Theory:** Subgraphs, Isomorphic and Homeomorphic Graphs, Paths, Connectivity, Labeled and Weighted Graphs, Complete, Regular, and Bipartite Graphs, Tree Graphs, Planar Graphs.	6	02, 03

ASSESSMENT PATTERN

Continuous Internal Evaluation (CIE 90 Marks)

Blooms Category	Test (Out of 45)	Assignments (15)	Quiz (15)	Co- curricular Activities (15)
Remember	05		5	Attendance 15
Understand	05			
Apply	10			
Analysis	8	7	10	
Evaluate	7	8		
Create	10			

Semester End Exam (SEE 60 Marks)

Blooms Category	Test (Out of 60)
Remember	10
Understand	10
Apply	10
Analysis	10
Evaluate	10
Create	10

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COURSE PLAN, SPECIFIC CONTENT, TEACHING LEARNING AND ASSESSMENT STRATEGY MAPPED WITH CLOS.

Week No.	Task Heading	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
1	**Set Theory**	Sets and elements, Subsets, Universal Sets, Empty Set, Disjoint Sets, Venn Diagrams	Lecture, Discussion	Presentation, Quiz	CLO 01, CLO 03
2	**Set Theory**	Set Operations, Algebra of Sets, Finite Sets, Counting Principle, Inclusion-Exclusion Principle	Lecture, Discussion	Quiz, Oral Presentation	CLO 01, CLO 03
3	**Set Theory**	Classes of Sets, Power Sets, Partitions, Mathematical Inductions	Lecture, Discussion	Group Assignment	CLO 01, CLO 03
4	**Relations**	Product Sets, Relations, Composition of Relations	Lecture, Discussion	Oral Presentation	CLO 01, CLO 03
5	**Relations**	Types of Relations	Lecture, Discussion	Written Assignment	CLO 01, CLO 03
6	**Relations**	Equivalence Relations, Partial Ordering Relations.	Lecture, Discussion	Written Assignment	CLO 01, CLO 03

COURSE PLAN, SPECIFIC CONTENT, TEACHING LEARNING AND ASSESSMENT STRATEGY MAPPED WITH CLOS.

Week No.	Task Heading	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
7	**Functions and Algorithms**	Functions, One-To-One, Onto and Invertible Functions	Lecture, Discussion	Presentation, Quiz	CLO 01, CLO 03
8	**Functions and Algorithms**	Mathematical Functions, Exponential and Logarithmic Functions	Lecture, Discussion	Quiz, Oral Presentation	CLO 01, CLO 03
9	**Logic and Propositional Calculus**	Propositions and Compound Statements, Basic Logical Operations	Lecture, Discussion	Oral Presentation, Group Assignment	CLO 01, CLO 03
10	**Logic and Propositional Calculus**	Proposition and Truth Tables, Tautologies and Contradictions	Lecture, Discussion	Oral Presentation	CLO 01, CLO 03
11	**Logic and Propositional Calculus**	Algebra of Propositions, Conditionals and Biconditional Statements	Lecture, Discussion	Written Assignment	CLO 01, CLO 03
12	**Boolean Algebra**	Logic Gates	Lecture, Discussion	Written Assignment	CLO 01, CLO 03

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COURSE PLAN, SPECIFIC CONTENT, TEACHING LEARNING AND ASSESSMENT STRATEGY MAPPED WITH CLOS.

Week No.	Task Heading	Topics	Teaching-Learning Strategy	Assessment Strategy	Corresponding CLO's
13	**Techniques of Counting**	Counting Principles, The Pigeonhole Principle	Lecture, Discussion	Presentation, Quiz	CLO2, CLO4
14	**Techniques of Counting**	Permutations, Combinations	Lecture, Discussion	Quiz, Oral Presentation	CLO2, CLO4
15	**Graph Theory**	Subgraphs, Isomorphic and Homeomorphic Graphs, Paths, Connectivity	Lecture, Discussion	Oral Presentation, Group Assignment	CLO2, CLO4
16	**Graph Theory**	Labeled and Weighted Graphs, Complete, Regular, and Bipartite Graphs	Lecture, Discussion	Oral Presentation	CLO2, CLO4
17	**Graph Theory**	Tree Graphs, Planar Graphs	Lecture, Discussion	Written Assignment	CLO1, CLO2

Set Theory

1st WEEK

Sets and elements, Subsets, Universal Sets, Empty Set, Disjoint Sets, Venn Diagrams

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SETS AND ELEMENTS

Sets: A set may be viewed as any well-defined collection of objects, called the elements or members of the set.

One usually uses capital letters, A, B, X, Y, . . . , to denote sets, and lowercase letters, a, b, x, y, . . ., to denote

elements of sets. Synonyms for “set” are “class,” “collection,” and “family.”

Membership in a set is denoted as follows:

$a \in S$ denotes that a belongs to a set S

$a, b \in S$ denotes that a and b belong to a set S

Here $\in$ is the symbol meaning “is an element of.”

We use $\notin$ to mean “is not an element of.”

SUBSETS

Suppose every element in a set A is also an element of a set B, that is, suppose $a \in A$ implies $a \in B$. Then

A is called a subset of B. We also say that A is contained in B or that B contains A. This relationship is written

$A \subseteq B$ or $B \supseteq A$

Two sets are equal if they both have the same elements or, equivalently, if each is contained in the other. That is: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

If A is not a subset of B, that is, if at least one element of A does not belong to B, we write $A \nsubseteq B$.

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UNIVERSAL SET, EMPTY SET

All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the **universal set**, denoted by U unless otherwise stated or implied.

Given a universal set U and a property P, there may not be any elements of U which have property P. For example, the following set has no elements:

$$S = \{x \mid x \text{ is a positive integer}, x^2 = 3\}$$

Such a set with no elements is called the empty set or null set and is denoted by $\emptyset$.

There is only one empty set. That is, if S and T are both empty, then $S = T$, since they have exactly the same elements, namely, none.

The empty set $\emptyset$ is also regarded as a subset of every other set. Thus we have the following simple result which we state formally. For any set A, we have $\emptyset \subseteq A \subseteq U$.

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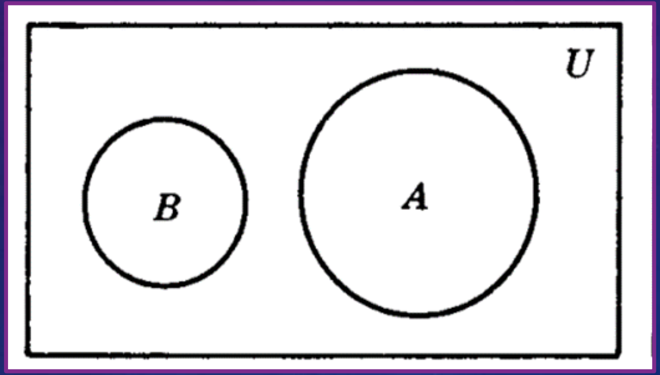
DISJOINT SETS

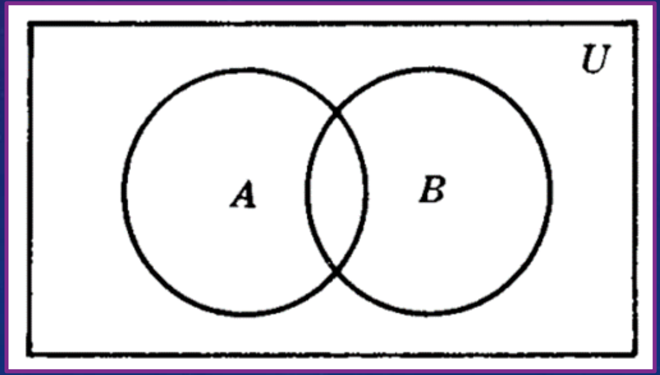
Two sets A and B are said to be disjoint if they have no elements in common. For example, suppose

$$A = \{1, 2\}, B = \{4, 5, 6\}, \text{and } C = \{5, 6, 7, 8\}$$

Then A and B are disjoint, and A and C are disjoint. But B and C are not disjoint since B and C have elements in common, e.g., 5 and 6. We note that if A and B are disjoint, then neither is a subset of the other (unless one is the empty set).



(1) A and B are disjoint



(2) A and B are not disjoint

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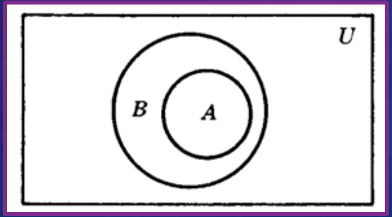
VENN DIAGRAMS

A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane. The universal set U is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle.

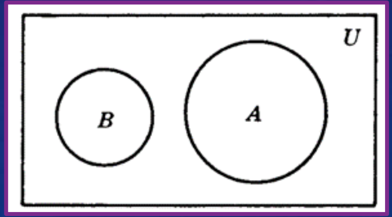
If A ⊆ B, then the disk representing A will be entirely within the disk representing B as in Fig. (a).

If A and B are disjoint, then the disk representing A will be separated from the disk representing B as in Fig (b).

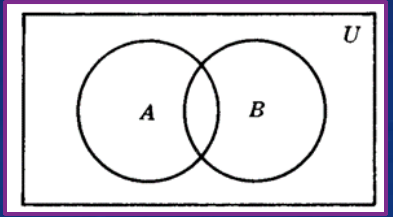
However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. (c).



(b) $A \subseteq B$



(b) A and B are disjoint



(c) A and B are not disjoint

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Set Theory

2nd WEEK

Set Operations, Finite, Countable/Uncountable Sets, Counting Principle

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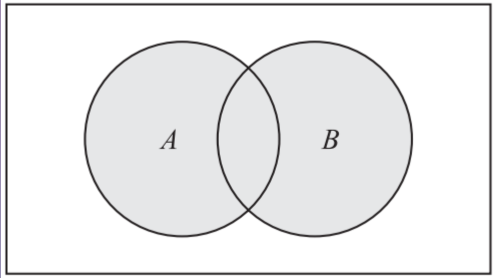
SET OPERATIONS

Union: The union of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

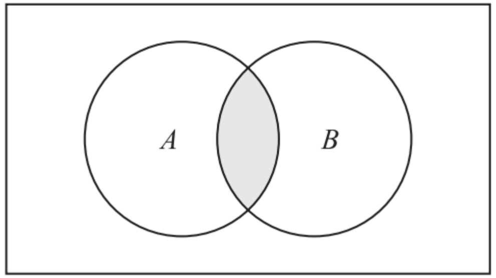
Here "or" is used in the sense of and/or.



(a) $A \cup B$ is shaded

Intersection: The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



(b) $A \cap B$ is shaded

Recall that sets A and B are said to be disjoint or nonintersecting if they have no elements in common or, using the definition of intersection, if $A \cap B = \emptyset$, the empty set. Suppose $S = A \cup B$ and $A \cap B = \emptyset$ Then S is called the disjoint union of A and B.

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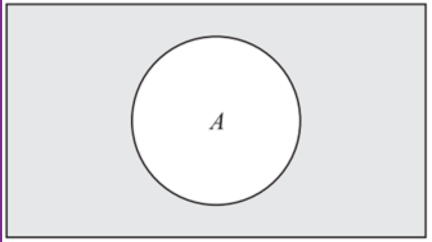
SET OPERATIONS

Complements: The complement of a set A, denoted by A^C, is the set of elements which belong to U but which do not belong to A. That is,

$$A^C = \{x \mid x \in U,\ x \notin A\}$$

Some texts denote the complement of A by A' or $\bar{A}$.

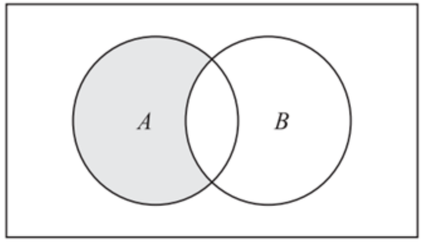


(a) A^C is shaded

Differences: The difference of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B. That is

$$A \backslash B = \{x \mid x \in A, x \notin B\}$$

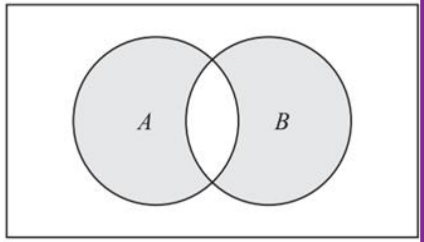
The set A\B is read "*A minus B*." Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$.



(b) $A \backslash B$ is shaded

Symmetric Differences: The symmetric difference of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both.

That is, $A \oplus B = (A \cup B)(A \cap B)$ *or* $A \oplus B = (A \backslash B) \cup (B \backslash A)$



(c) $A \oplus B$ is shaded

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FINITE, COUNTABLE/UNCOUNTABLE SETS

Sets can be finite or infinite. A set S is said to be finite if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is infinite.

EXAMPLE 1. The set A of the letters of the English alphabet and the set D of the days of the week are finite sets. Specifically, A has 26 elements and D has 7 elements.

EXAMPLE 2. Let E be the set of even positive integers, and let I be the unit interval, that is, $E = \{2, 4, 6, \ldots\}$ and $I = [0, 1] = \{x \mid 0 \leq x \leq 1\}$

Then both E and I are infinite.

A set S is countable if S is finite or if the elements of S can be arranged as a sequence, in which case S is said to be countably infinite; otherwise S is said to be uncountable. The above set E of even integers is countably infinite, whereas one can prove that the unit interval $I = [0, 1]$ is uncountable.

COUNTING PRINCIPLE

The notation $n(S)$ or $|S|$ will denote the number of elements in a set S. (Some texts use $\#(S)$ or $card(S)$ instead of $n(S)$. Thus $n(A) = 26$, where A is the letters in the English alphabet, and $n(D) = 7$, where D is the days of the week.

Also $n(\emptyset) = 0$ since the empty set has no elements.

1) If S is the disjoint union of finite sets A and B. Then S is finite and
$$n(S) = n(A) + n(B)$$
2) If A and B be finite sets. Then $n(A \backslash B) = n(A) - n(A \cap B)$
3) If A be a subset of a finite universal set U. Then $n(A^C) = n(U) - n(A)$

INCLUSION/EXCLUSION PRINCIPLE

Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to obtain a similar formula for three sets:

Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

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INCLUSION/EXCLUSION PRINCIPLE

Example 01: Each student in Liberal Arts at some college has a mathematics requirement A and a science requirement B. A poll of 140 sophomore students shows that:

60 completed A,

45 completed B,

20 completed both A and B.

Use a Venn diagram to find the number of students who have completed:

(a) At least one of A and B;

(b) exactly one of A or B;

(c) neither A nor B.

INCLUSION/EXCLUSION PRINCIPLE

Solution to Example-01: Translating the above data into set notation yields:

$n(A) = 60$, $n(B) = 45$, $n(A \cap B) = 20$, $n(U) = 140$

Draw a Venn diagram of sets A and B as in Fig. 01, assign numbers to the four regions as follows:

20 completed both A and B, so $n(A \cap B) = 20$.

$60 - 20 = 40$ completed A but not B, so $n(A \backslash B) = 40$.

$45 - 20 = 25$ completed B but not A, so $n(B \backslash A) = 25$.

$140 - 20 - 40 - 25 = 55$ completed neither A nor B.

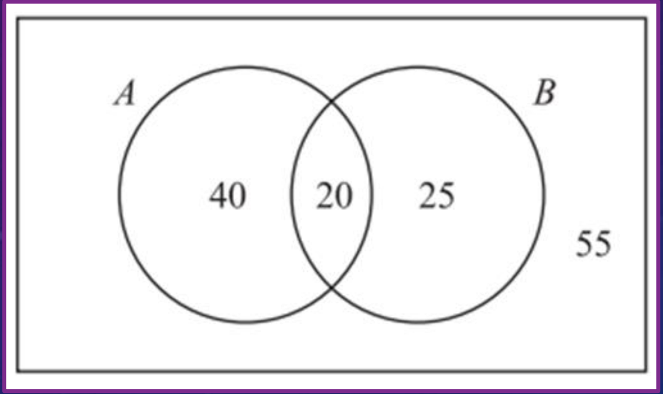

Fig. 01

By the Venn diagram, as in Fig. 01:

(a) $20 + 40 + 25 = 85$ completed A or B. Alternately, by the Inclusion–Exclusion Principle:

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 60 + 45 - 20 = 85$

(b) $40 + 25 = 65$ completed exactly one requirement. That is, $n(A \oplus B) = 65$.

(c) 55 completed neither requirement, i.e.

$n(A^C \cap B^C) = n[(A \cup B)^C] = 140 - 85 = 55.$

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INCLUSION/EXCLUSION PRINCIPLE

Example 02: In a survey of 120 people, it was found that:

65 read Newsweek magazine,
45 read Time,
42 read Fortune,
20 read both Newsweek and Time,
25 read both Newsweek and Fortune,
15 read both Time and Fortune,
8 read all three magazines.

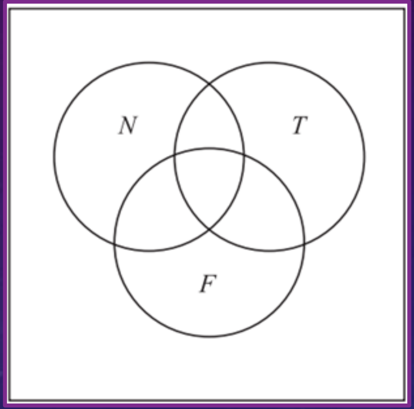

Fig. 02(a)

a) Find the number of people who read at least one of the three magazines.

b) Fill in the correct number of people in each of the eight regions of the Venn diagram in Fig. 02(a) where N, T, and F denote the set of people who read Newsweek, Time, and Fortune, respectively.

c) Find the number of people who read exactly one magazine.

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INCLUSION/EXCLUSION PRINCIPLE

Solution to Example-02:

(a) We want to $find\ n(N \cup T \cup F)$. By Inclusion–Exclusion Principle,

$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$

$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$

(b) The required Venn diagram in Fig. 02(b) is obtained as follows:

8 read all three magazines,

$20 - 8 = 12$ read Newsweek and Time but not all three magazines,

$25 - 8 = 17$ read Newsweek and Fortune but not all three magazines,

$15 - 8 = 7$ read Time and Fortune but not all three magazines,

$65 - 12 - 8 - 17 = 28$ read only Newsweek,

$45 - 12 - 8 - 7 = 18$ read only Time,

$42 - 17 - 8 - 7 = 10$ read only Fortune,

$120 - 100 = 20$ read no magazine at all.

(c) $28 + 18 + 10 = 56$ read exactly one of the magazines.

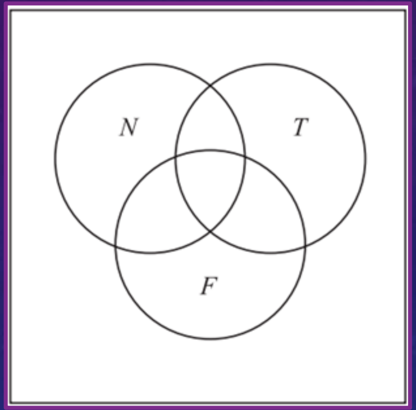

Fig. 02(a)

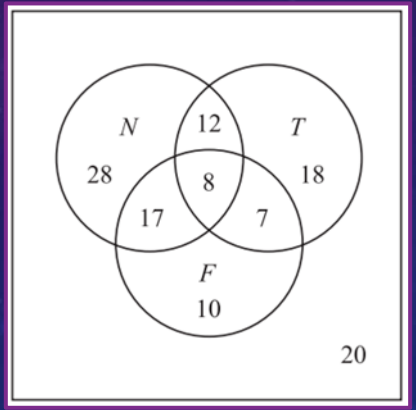

Fig. 02(b)

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Set Theory

3rd WEEK

Classes of Sets, Power Sets, Partitions, Mathematical Inductions

Page No: 27 - 31

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CLASSES OF SETS

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a set of sets. Whenever such a situation occurs, to avoid confusion, we will speak of a class of sets or collection of sets rather than a set of sets. If we wish to consider some of the sets in a given class of sets, then we speak of subclass or subcollection.

Example – 01: Suppose $S = \{1, 2, 3, 4\}$.

(a) Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

That is, the elements of A are the sets $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$, and $\{2, 3, 4\}$.

(b) Let B be the class of subsets of S, each which contains 2 and two other elements of S. Then

$$B = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of B are the sets $\{1, 2, 3\}, \{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

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POWER SETS

For a given set S , we may speak of the class of all subsets of S. This class is called the power set of S , and will be denoted by $P(S)$. If S is finite, then so is $P(S)$. In fact, the number of elements in $P(S)$ is 2 raised to the power $n(S)$. That is,

$$n(P\ (S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.)

Example: Suppose $S = \{1, 2, 3\}$. Then

$$P\ (S) = [\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S]$$

Note that the empty set $\emptyset$ belongs to $P(S)$ since $\emptyset$ is a subset of S. Similarly, S belongs to $P(S)$.

As expected from the above remark, $P(S)$ has $2^3 = 8$ elements.

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PARTITIONS

Let S be a nonempty set. A partition of S is a subdivision of S into nonoverlapping, nonempty subsets.

Precisely, a partition of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i

(ii) The sets of {Ai} are mutually disjoint; that is, if

$$A_j \neq A_k \text{ then } A_j \cap A_k = \emptyset$$

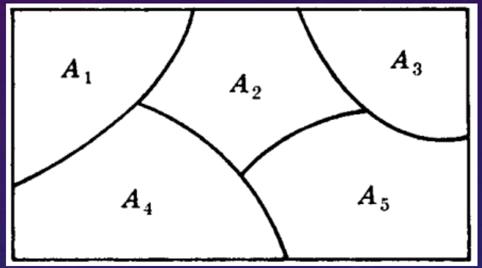

Fig. 03

The subsets in a partition are called cells. Fig. 03 is a Venn diagram of a partition of the rectangular set S of points into five cells, $A1, A2, A3, A4, A5$

<u>Example:</u> Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$

(ii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$

(iii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. On the other hand, (iii) is a partition of S.

MATHEMATICAL INDUCTIONS

Let P be a proposition defined on the positive integers N; that is, $P(n)$ is either true or false for each $n \in N$. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(k+1)$ is true whenever $P(k)$ is true.

Then P is true for every positive integer $n \in N$.

Example - 01: Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n): 1+3+5+\cdots+(2n-1)=n^2$$

(The k^{th} odd number is $2k-1$, and the next odd number is $2k+1$.) Observe that $P(n)$ is true for $n=1$; namely,

$$P(1)=1^2$$

Assuming $P(k)$ is true, we add $2k+1$ to both sides of $P(k)$, obtaining

$$1+3+5+\cdots+(2k-1)+(2k+1)-k^2+(2k+1)=(k+1)^2$$

which is $P(k+1)$. In other words, $P(k+1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.

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MATHEMATICAL INDUCTIONS

<u>Example - 02:</u> Prove the proposition $P(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is, $P(n) = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

<u>Solution:</u> The proposition holds for $n = 1$ since,

$$P(1): 1 = \frac{1}{2}(1)(1+1)$$

Assuming P (k) is true, we add k + 1 to both sides of $P(k)$, obtaining

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{1}{2}k(k+1) + (k+1)$$

$$= \frac{1}{2}[k(k+1) + 2(k+1)]$$

$$= \frac{1}{2}[(k+1) + (k+2)]$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all n.

ASSIGNMENT – 01

1. There are 400 students is a University. Out of them 345 students taken a course in calculus, 212 taken a course in discrete mathematics, and 188 taken courses in both calculus and discrete mathematics. Find the number of students who have taken a course in:

   a) either calculus or discrete mathematics;

   b) exactly one of the courses;

   c) neither calculus nor discrete mathematics.

2. In a university 2092 students have taken at least one of Spanish, French, and Russian language courses. Of these 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Also, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. How many students have taken a course in all three languages?

3. Prove $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ using mathematical induction method

4. Prove $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ when n is a non-negative integer.

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Relations

4th WEEK

Product Sets, Relations, Composition of Relations

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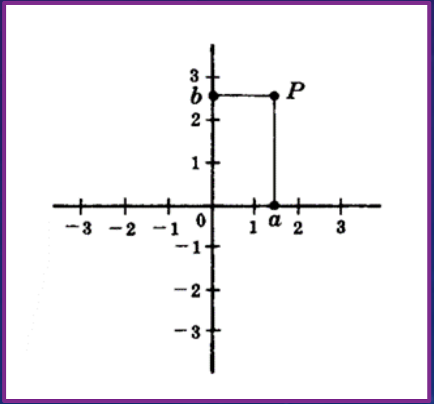
PRODUCT SETS

An **Ordered Pair** of elements a and b, where a is designated as the first element and b as the second element, is denoted by (a, b).

Product Sets: Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the product, or Cartesian product, of A and B. A short designation of this product is $A \times B$, which is read "$A\ cross\ B$." By definition, $A \times B = \{(a, b) | a \in A\ and\ b \in B\}$

One frequently writes A^2 instead of $A \times A$.



Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Note: If R denotes the set of real numbers and so $R^2 = R \times R$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of R^2 as points in the plane. Here each point P represents an ordered pair (a, b) of real numbers and vice versa.

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PRODUCT SETS

Note: In fact, $n(A \times B) = n(A)n(B)$ for any finite sets A and B. This follows from the observation that, for an ordered pair (a, b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

The idea of a product of sets can be extended to any finite number of sets. For any sets $A_1, A_2, \dots, A_n$, the set of all ordered $n - tuples$ $(a_1, a_2, \dots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$ is called the product of the sets $A_1, \dots, A_n$ and is denoted by

$$A_1 \times A_2 \times \cdots \times A_n \; or \; \prod_{i=1}^{n} A_1$$

Just as we write A^2 instead of $A \times A$, so we write A_n instead of $A \times A \times \cdots \times A$, where there are n factors all equal to A. For example, $R^3 = R \times R \times R$ denotes the usual three-dimensional space.

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RELATIONS

Let A and B be sets. A binary relation or, simply, relation from A to B is a subset of $A \times B$. Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

1) $(a, b) \in R$; we then say "$a\ is\ R - related\ to\ b$", written aRb.

2) $(a, b) \notin R$; we then say "$a\ is\ not\ R - related\ to\ b$", written $a\not{R}b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A. The domain of a relation R is the set of all first elements of the ordered pairs which belong to R, and the range is the set of second elements.

Example: $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry, 1Rz, 3Ry, but\ 1\not{R}x, 2\not{R}x, 2\not{R}y, 2\not{R}z, 3\not{R}x, 3\not{R}z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

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INVERSE RELATIONS

Let R be any relation from a set A to a set B. The inverse of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b,a)|(a,b) \in R\}$$

Example: let $A = \{1,2,3\}$ and $B = \{x,y,z\}$. Then the inverse of

$$R = \{(1,y),(1,z),(3,y)\} \quad \text{is} \quad R^{-1} = \{(y,1),(z,1),(y,3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

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COMPOSITION OF RELATIONS

Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by:

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc.$$

That is ,

$$R \circ S = \{(a, c) | \, there\ exists\ b \in B\ for\ which\ (a, b) \in R\ and\ (b, c) \in S\}$$

The relation $R \circ S$ is called the composition of R and S; it is sometimes denoted simply by RS.

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself, is always defined. Also, $R \circ R$ is sometimes denoted by R^2.

Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

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COMPOSITION OF RELATIONS

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} \text{ and } S = \{(b, x), (b, z), (c, y), (d, z)\}$$

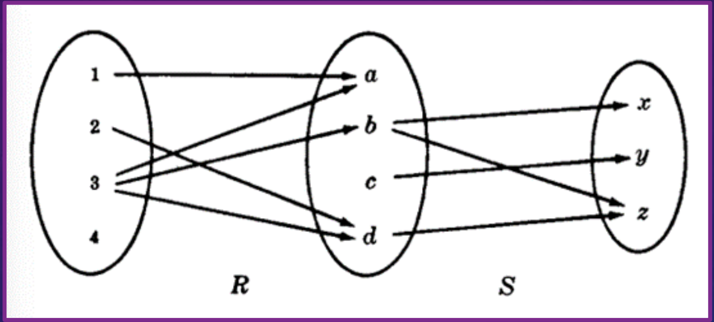

Fig. 04

Consider the arrow diagrams of R and S as in Fig. 04.

Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element 2 ∈ A to the element z ∈ C.

Thus: 2(R ◦ S)z since 2Rd and dSz Similarly there is a path from 3 to x and a path from 3 to z. Hence 3(R◦S)x and 3(R◦S)z. No other element of A is connected to an element of C. Accordingly, R ◦ S = {(2, z), (3, x), (3, z)}. We get that, composition of relations is associative.

Associative Law: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then (R ◦ S) ◦ T = R ◦ (S ◦ T)

Relations

5th WEEK

Types of Relations

Page No: 41 - 42

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TYPES OF RELATIONS

Reflexive Relations: A relation R on a set A is reflexive if aRa for every a ∈ A, that is, if (a, a) ∈ R for every a ∈ A. Thus R is not reflexive if there exists a ∈ A such that (a, a) ∉ R.

Symmetric Relations: A relation R on a set A issymmetric if whenever aRb then bRa, that is, if whenever(a, b) ∈ R then (b, a) ∈ R. Thus R is not symmetric if there exists a, b ∈ A such that (a, b) ∈ R but (b, a) ∉ R.

Antisymmetric Relations: A relation R on a set A is antisymmetric if whenever aRb and bRa then a = b, that is, if a = b and aRb then bRa. Thus R is not antisymmetric if there exist distinct elements a and b in A such that aRb and bRa.

Transitive Relations: A relation R on a set A is transitive if whenever aRb and bRc then aRc, that is, if whenever (a, b), (b, c) ∈ R then (a, c) ∈ R. Thus R is not transitive if there exist a, b, c ∈ R such that (a, b), (b, c) ∈ R but (a, c) ∉ R.

TYPES OF RELATIONS

Example-01: Consider the following five relations on the set A = {1, 2, 3, 4}:

$R_1 = \{(1,1),(1,2),(2,3),(1,3),(4,4)\}$ $R_2 = \{(1,1)(1,2),(2,1),(2,2),(3,3),(4,4)\}$

$R_3 = \{(1,3),(2,1)\}$ $R_4 = \emptyset$, the empty relation $R_5 = A \times A$, the universal relation

Question: Determine which of the relations are Reflexive, Symmetric, Antisymmetric or Transitive?

Reflexive: Since A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs (1, 1), (2, 2), (3, 3), and (4, 4). Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, (2, 2) does not belong to any of them.

Symmetric: R_1 is not symmetric since (1, 2) ∈ R_1 but (2, 1) ∉ R_1. R_3 is not symmetric since (1, 3) ∈ R_3 but (3, 1) ∉ R_3. The other relations are symmetric.

Antisymmetric: R_2 is not antisymmetric since (1, 2) and (2, 1) belong to R_2, but 1 ≠ 2. Similarly, the universal relation R_3

is not antisymmetric. All the other relations are antisymmetric.

Transitive: The relation R_3 is not transitive since (2, 1),(1, 3) ∈ R_3 but (2, 3) ∉ R_3. All the other relations are transitive.

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Relations

6th WEEK

Equivalence Relations,
Partial Ordering Relations.

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EQUIVALENCE RELATIONS

Consider a nonempty set S. A relation R on S is an equivalence relation if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, aRa.

(2) If aRb, then bRa.

(3) If aRb and bRc, then aRc.

Example: The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$

PARTIAL ORDERING RELATIONS

A relation R on a set S is called a partial ordering or a partial order of S if R is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set or poset. That is, R is a partial ordering relation on S if it has the following three properties:

(1) For every $a \in S$, aRa.

(2) (2) If aRb, bRa then a = b.

(3) (3) If aRb and bRc, then aRc.

Example: The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set A.

(2) (2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(3) (3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

ASSIGNMENT – 02

1) Let A = {1, 2, 3}, B = {a, b, c}, and C = {x, y,z}. Consider the following relations R and S from A to B and from B to C, respectively.

$$R = \{(1, b), (2, a), (2, c)\} \text{ and } S = \{(a, y), (b, x), (c, y), (c, z)\}$$

   a) Find the composition relation R◦S.

   b) Find the matrices M_R, M_S, and $M_R \circ S$ of the respective relations R, S, and R◦S, and compare $M_R \circ S$ to the product $M_R M_S$.

2) Consider the following five relations on the set A = {1, 2, 3}:

R = {(1, 1), (1, 2), (1, 3), (3, 3)}, Ø = empty relation

S = {(1, 1)(1, 2), (2, 1)(2, 2), (3, 3)}, A × A = universal relation

T = {(1, 1), (1, 2), (2, 2), (2, 3)}

Determine whether or not each of the above relations on A is:

(a) reflexive; (b) symmetric; (c) transitive; (d) antisymmetric.

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ASSIGNMENT – 02

3) Prove: (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$; (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

4) Let R be the following equivalence relation on the set A = {1, 2, 3, 4, 5, 6}:

   R = {(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)}

   Find the partition of A induced by R, i.e., find the equivalence classes of R.

5) Consider the set Z of integers. Define aRb by $b = a^r$ for some positive integer r.

   Show that R is a partial order on Z, that is, show that R is:

   (a) reflexive; (b) antisymmetric; (c) transitive.

Functions and Algorithms

7th WEEK

Functions, One-To-One, Onto and Invertible Functions

Page No: 49 - 51

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FUNCTIONS

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a $function$ from A into B. The set A is called the $domain$ of the function, and the set B is called the target set or $codomain$. Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: A \rightarrow B$$

which is read: "$f\ is\ a\ function\ from\ A\ into\ B$," or "$f\ takes\ (or\ maps)\ A\ into\ B$." If a ∈ A, then $f(a)$ (read: "$f\ of\ a$") denotes the unique element of B which f assigns to a; it is called the image of a under f, or the value of f at a. The set of all image values is called the $range$ or $image$ of f. The image of $f : A \rightarrow B$ is denoted by $Ran(f), Im(f)\ or\ f(A)$.

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ONE-TO-ONE, ONTO AND INVERTIBLE FUNCTIONS

A function $f : A \rightarrow B$ is said to be one-to-one (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is one-to-one if f (a) = f (a′) implies a = a ′ .

A function $f: A \rightarrow B$ is said to be an onto function if each element of B is the image of some element of A. In other words, f : A → B is onto if the image of f is the entire codomain, i.e., if f (A) = B. In such a case we say that f is a function from A onto B or that f maps A onto B.

A function $f: A \rightarrow B$ is invertible if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function.

Theorem: A function f: A → B is invertible if and only if f is both one-to-one and onto.

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ONE-TO-ONE, ONTO AND INVERTIBLE FUNCTIONS

Example: Consider the functions $f_1: A \rightarrow B, f_2: B \rightarrow C, f_3: C \rightarrow D$ and $f_4: D \rightarrow E$ defined by the diagram of Fig. 05

f_1 and f_2 is one-to-one since no element of B is the image of more than one element of A.

However, f_3 and f_4 is not one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$

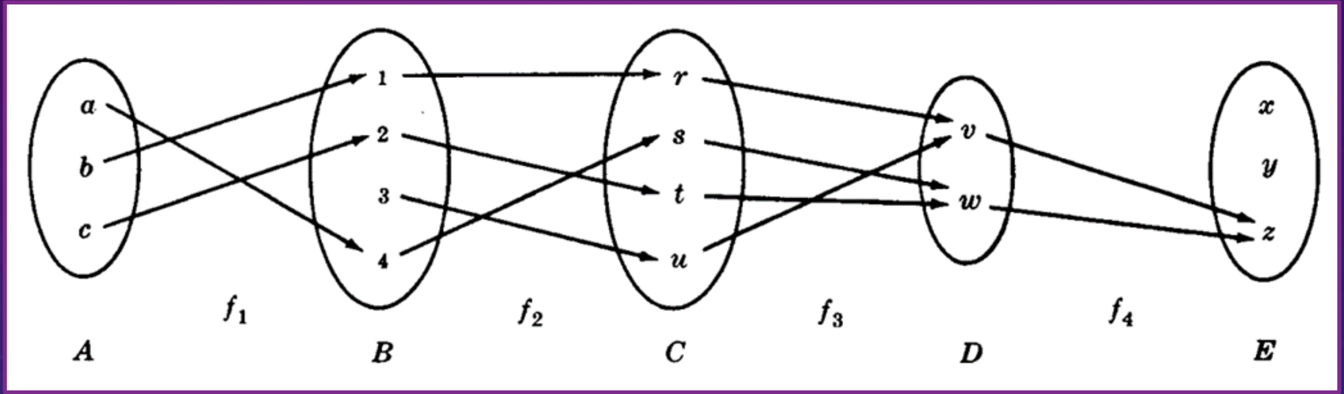

Fig. 05

f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, $f_2(B) = C$ and $f_3(C) = D$.

On the other hand, f_1 is not onto since 3 ∈ B is not the image under f_4 of any element of A and f_4 is not onto since x ∈ E is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto.

However, f_2 is both one-to-one and onto, i.e., is a one-to-one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Functions and Algorithms

8th WEEK

Mathematical Functions, Exponential and Logarithmic Functions

Page No: 53 - 57

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MATHEMATICAL FUNCTIONS

Floor and Ceiling Functions: Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically,

$\lfloor x \rfloor$, called the floor of x, denotes the greatest integer that does not exceed x.

$\lceil x \rceil$, called the ceiling of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.

For example,

$\lfloor 3.14 \rfloor = 3$, $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor -8.5 \rfloor = -9$, $\lfloor 7 \rfloor = 7$, $\lfloor -4 \rfloor = -4$,

$\lceil 3.14 \rceil = 4$, $\lceil \sqrt{5} \rceil = 3$, $\lceil -8.5 \rceil = -8$, $\lceil 7 \rceil = 7$, $\lceil -4 \rceil = -4$

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MATHEMATICAL FUNCTIONS

Integer and Absolute Value Functions: Let x be any real number. The integer value of x, written INT(x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus

$\mathrm{INT}(3.14) = 3$, $\mathrm{INT}(\sqrt{5}) = 2$, $\mathrm{INT}(-8.5) = -8$, $\mathrm{INT}(7) = 7$

Observe that $\mathrm{INT}(x) = \lfloor x \rfloor$ or $\mathrm{INT}(x) = \lceil x \rceil$ according to whether x is positive or negative.

The absolute value of the real number x, written ABS(x) or $|x|$, is defined as the greater of x or $-x$. Hence $\mathrm{ABS}(0) = 0$, and, for $x = 0$, $\mathrm{ABS}(x) = x$ or $\mathrm{ABS}(x) = -x$, depending on whether x is positive or negative. Thus

$|-15| = 15$, $|7| = 7$, $|-3.33| = 3.33$, $|4.44| = 4.44$, $|-0.075| = 0.075$

We note that $|x| = |-x|$ and, for $x \neq 0$, $|x|$ is positive.

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MATHEMATICAL FUNCTIONS

Remainder Function and Modular Arithmetic: Let k be any integer and let M be a positive integer. Then $k(mod\ M)$ (read: k modulo M) will denote the integer remainder when k is divided by M. More exactly, k (mod M) is the unique integer r such that $k = Mq + r$ where $0 \leq r < M$

When k is positive, simply divide k by M to obtain the remainder r. Thus

25 (mod 7) = 4, 25 (mod 5) = 0, 35 (mod 11) = 2, 3 (mod 8) = 3

If k is negative, divide $|k|$ by M to obtain a remainder r' ; then $k(mod\ M) = M - r'$ when $r' \neq 0$. Thus $-26(mod\ 7) = 7 - 5 = 2,$ $-371(mod\ 8) = 8 - 3 = 5,$ $-39(mod\ 3) = 0$

The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows: $a \equiv b\ (mod\ M)$ if any only if M divides $b - a$

M is called the modulus, and $a \equiv b\ (mod\ M)$ is read "$a\ is\ congruent\ to\ b\ modulo\ M$".

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EXPONENTIAL FUNCTIONS

The following definitions for integer exponents (where m is a positive integer):

$$a^m = a \cdot a \cdots a(m\ times),\ a^0 = 1,\ a^{-m} = \frac{1}{a^m}$$

Exponents are extended to include all rational numbers by defining, for any rational number m/n, $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

For example, $2^4 = 16,\ 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$, $125^{2/3} = 5^2 = 25$

Exponents are extended to include all real numbers by defining, for any real number x,

$$a^x = \lim_{r \to x} a^r$$, where r is a rational number

Accordingly, the exponential function $f(x) = a^x$ is defined for all real numbers.

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LOGARITHMIC FUNCTIONS

Logarithms are related to exponents as follows. Let b be a positive number. The logarithm of any positive number x to be the base b, written

$$\log_b x$$

represents the exponent to which b must be raised to obtain x. That is, $y = \log_b x$ and $b^y = x$ are equivalent statements. Accordingly,

$\log_2 8 = 3$ since $2^3 = 8$; $\log_{10} 100 = 2$ since $10^2 = 100$

$\log_2 64 = 6$ since $2^6 = 64$; $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$

The logarithm of a negative number and the logarithm of 0 are not defined.

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Logic and Propositional Calculus

9th WEEK

Propositions and Compound Statements, Basic Logical Operations

Page No: 59 - 63

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PROPOSITIONS AND COMPOUND STATEMENTS

A proposition (or statement) is a declarative statement which is true or false, but not both. Consider, for example, the following six sentences:

(i) Ice floats in water. (iii) 2 + 2 = 4 (v) Where are you going?

(ii) China is in Europe. (iv) 2 + 2 = 5 (vi) Do your homework.

The first four are propositions, the last two are not. Also, (i) and (iii) are true, but (ii) and (iv) are false.

Compound Proposition: Many propositions are composite, such composite propositions are called compound propositions. A proposition is said to be primitive if it cannot be broken down into simpler propositions, that is, if it is not composite.

For example, the above propositions (i) through (iv) are primitive propositions. On the other hand, the following two propositions are composite:

"Roses are red and violets are blue." and "John is smart or he studies every night."

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BASIC LOGICAL OPERATIONS

There are three basic logical operations of conjunction, disjunction, and negation which correspond, respectively, to the English words “and,” “or,” and “not.”

Conjunction, p ∧ q: Any two propositions can be combined by the word “and” to form a compound proposition called the conjunction of the original propositions. Symbolically, $p \wedge q$ read “p and q,” denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q.

Specifically: If p and q are true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(*a*) “*p* and *q*”

Example: Consider the following four statements:

(i) Ice floats in water and 2 + 2 = 4. (iii) China is in Europe and 2 + 2 = 4.

(ii) Ice floats in water and 2 + 2 = 5. (iv) China is in Europe and 2 + 2 = 5.

Only the first statement is true. Each of the others is false since at least one of its substatements is false.

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BASIC LOGICAL OPERATIONS

Disjunction, p ∨ q: Any two propositions can be combined by the word "or" to form a compound proposition called the disjunction of the original propositions. Symbolically, p ∨ q read "p or q," denotes the disjunction of p and q. The truth value of p ∨ q depends only on the truth values of p and q as follows: If p and q are false, then p ∨ q is false; otherwise p ∨ q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

Example: Consider the following four statements:

(i) Ice floats in water or 2 + 2 = 4. (iii) China is in Europe or 2 + 2 = 4.

(ii) Ice floats in water or 2 + 2 = 5. (iv) China is in Europe or 2 + 2 = 5.

Only the last statement (iv) is false. Each of the others is true since at least one of its sub-statements is true.

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BASIC LOGICAL OPERATIONS

Negation, ¬p: Given any proposition p, another proposition, called the negation of p, can be formed by writing "It is not true that ..." or "It is false that ..." before p or, if possible, by inserting in p the word "not." Symbolically, the negation of p, read "not p," is denoted by ¬p The truth value of ¬p depends on the truth value of p as follows: If p is true, then ¬p is false; and if p is false, then ¬p is true.

Example: Consider the following six statements:

(a_1) Ice floats in water. (a_2) It is false that ice floats in water. (a_3) Ice does not float in water.

(b_1) $2 + 2 = 5$ (b_2) It is false that $2 + 2 = 5$. (b_3) $2 + 2 \neq 5$

Then (a_2) and (a_3) are each the negation of (a_1); and (b_2) and (b_3) are each the negation of (b_1).

Since (a_1) is true, (a_2) and (a_3) are false;

and since (b_1) is false, (b_2) and (b_3) are true.

p	$\neg p$
T	F
F	T

(c) "not p"

BASIC LOGICAL OPERATIONS

An Important Remark: The logical notation for the connectives “and,” “or,” and “not” is not completely standardized. For example, some texts use:

$p \,\&\, q, p \cdot q$ or pq	for	$p \wedge q$
$p + q$	for	$p \vee q$
$p', \bar{p}$ or $\sim p$	for	$\neg p$

Logic and Propositional Calculus

10th WEEK

Proposition and Truth Tables, Tautologies and Contradictions

Page No: 65 - 67

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PROPOSITION AND TRUTH TABLES

Let $P(p, q, \ldots)$ denote an expression constructed from logical variables $p, q, \ldots$, which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\neg$ Such an expression $P(p, q, \ldots)$ will be called a $proposition$.

The conventional letters used for propositional variables are $p, q, r, s, \ldots \ldots \ldots$ The truth value of a proposition is $true$, denoted by T, if it is a $true\ proposition$, and the truth value of a proposition is $false$, denoted by F, if it is a $false\ proposition$.

For Example: The fig. 06 shows a truth table.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Fig. 06

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TAUTOLOGIES AND CONTRADICTIONS

Tautology: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

Contradiction: A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example: The example shown in fig. 07(a) is a tautology; and fig. 07(b) is a contradiction by using truth tables.

Example: Show that the conditional statement $[\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology by using truth tables.

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

From truth table we can say that $[\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology.

Fig. 07 (a)

Example: Show that the conditional statement $\neg p \wedge [(p \vee \neg q) \wedge q]$ is a fallacy/contradiction by using truth tables.

p	q	$\neg p$	$\neg q$	$(p \vee \neg q)$	$(p \vee \neg q) \wedge q$	$\neg p \wedge [(p \vee \neg q) \wedge q]$
T	T	F	F	T	T	F
T	F	F	T	T	F	F
F	T	T	F	F	F	F
F	F	T	T	T	F	F

According to the truth tables we conclude that $\neg p \wedge [(p \vee \neg q) \wedge q]$ is a fallacy/ contradiction.

Fig. 07 (b)

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TAUTOLOGIES AND CONTRADICTIONS

Exercise :

1. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology or not.
2. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology or not.
3. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.
4. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.

Logical Equivalence: Two propositions P (p, q, . . .) and Q(p, q, . . .) are said to be logically equivalent, or simply equivalent or equal, denoted by

$$P\,(p, q, \dots) \equiv Q(p, q, \dots)$$

if they have identical truth tables. In other words, the propositions are logically equivalent.

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Logic and Propositional Calculus

11th WEEK

Algebra of Propositions, Conditionals and Biconditional Statements

Page No: 69 - 73

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ALGEBRA OF PROPOSITIONS

Propositions satisfy various laws which are listed in Table-01. (In this table, T and F are restricted to the truth values "True" and "False," respectively.) We state this result formally: Propositions satisfy the laws of Table-01

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Table-01

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CONDITIONALS AND BICONDITIONAL

Conditionals: Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

The conditional $p \to q$ is frequently read "p implies q" or "p only if q."

Biconditionals: Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

CONDITIONALS AND BICONDITIONAL

Exercise: Let p, q, and r be the propositions

p: Grizzly bears have been seen in the area. q: Hiking is safe on the trail. r: Berries are ripe along the trail.

Write these propositions using p, q, and r and logical connectives (including negations).

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.

b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.

e) For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.

f) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

Answer: a) $r \wedge \neg p$ b) $\neg p \wedge q \wedge r$ c) $r \rightarrow (q \leftrightarrow \neg p)$ d) $\neg q \wedge \neg p \wedge r$

e) $(q \rightarrow (\neg r \wedge \neg p)) \wedge \neg((\neg r \wedge \neg p) \rightarrow q)$ f) $(p \wedge r) \rightarrow \neg q$

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CONDITIONALS AND BICONDITIONAL

Exercise: Construct a truth table for each of these compound propositions.

a) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

b) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

c) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$

d) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

e) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

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CONDITIONALS AND BICONDITIONAL

Exercise:

1) Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.
2) Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.
3) Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.
4) Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.
5) Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.
6) Show that $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are logically equivalent.
7) Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.
8) Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.
9) Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.
10) Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.
11) Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.
12) Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.

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Boolean Algebra

12th WEEK

Logic Gates

Page No: 75 - 79

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LOGIC GATES

The inverter, or NOT gate, takes an input bit p, and produces as output ¬p. The OR gate takes two input signals p and q, each a bit, and produces as output the signal p ∨ q. Finally, the AND gate takes two input signals p and q, each a bit, and produces as output the signal p ∧ q.

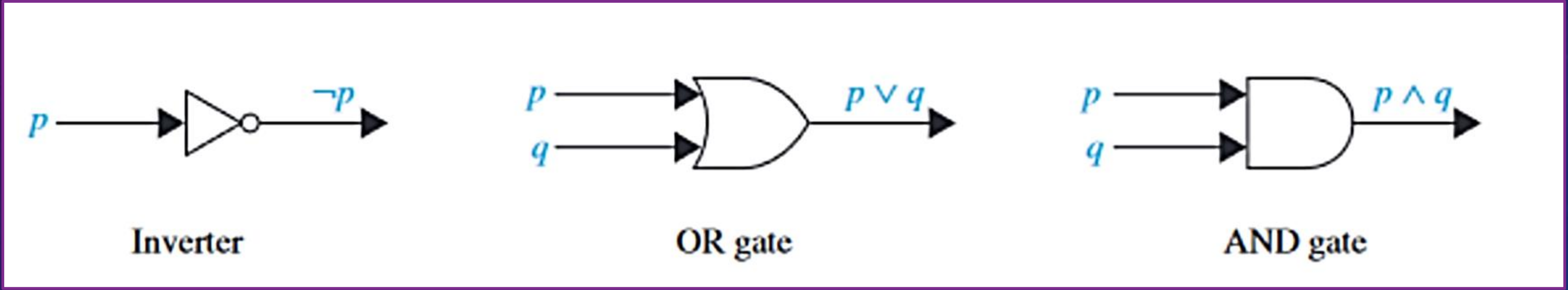

Fig. 08 : Basic logic gates

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LOGIC GATES

Question-01: Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$ when given input bits p, q, and r.

Solution: To construct the desired circuit, we build separate circuits for p ∨ ¬r and for ¬p ∨(q ∨ ¬r) and combine them using an AND gate. To construct a circuit for p ∨ ¬r, we use an inverter to produce ¬r from the input r. Then, we use an OR gate to combine p and ¬r. To build a circuit for ¬p ∨ (q ∨ ¬r), we first use an inverter to obtain ¬r. Then we use an OR gate with inputs q and ¬r to obtain q ∨ ¬r. Finally, we use another inverter and an OR gate to get ¬p ∨ (q ∨ ¬r) from the inputs p and q ∨ ¬r. To complete the construction, we employ a final AND gate, with inputs p ∨ ¬r and ¬p ∨(q ∨ ¬r). The resulting circuit is displayed in Figure 09.

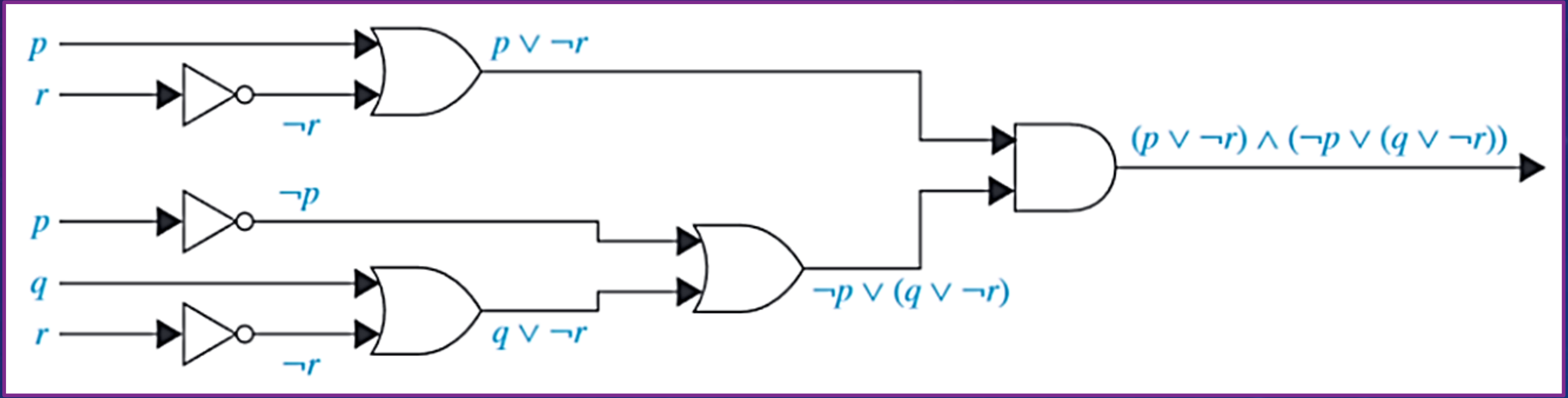

Fig. 09 : The circuit for (p ∨ ¬r) ∧ (¬p ∨ (q ∨ ¬r)).

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LOGIC GATES

Question-02: Determine the output for the combinatorial circuit in Figure 10.

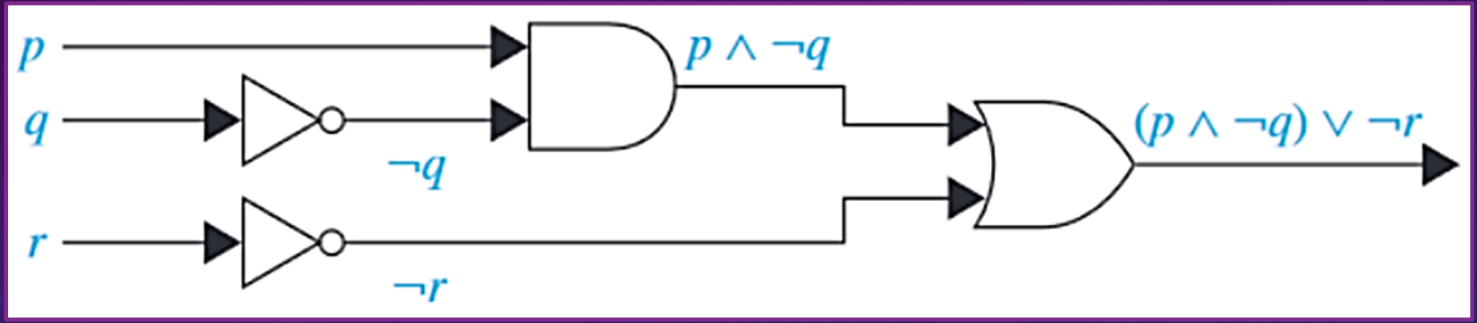

Fig. 10 : A combinatorial circuit

Solution: In Figure 10 we display the output of each logic gate in the circuit. We see that the AND gate takes input of p and ¬q, the output of the inverter with input q, and produces p ∧ ¬q.

Next, we note that the OR gate takes input p ∧ ¬q and ¬r, the output of the inverter with input r, and produces the final output (p ∧ ¬q) ∨ ¬r.

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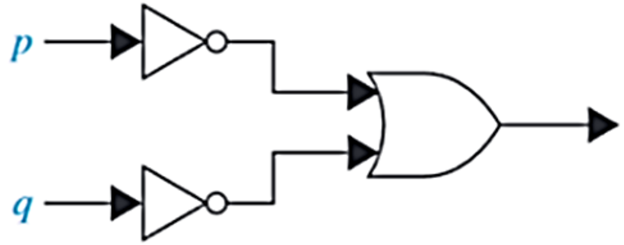
LOGIC GATES

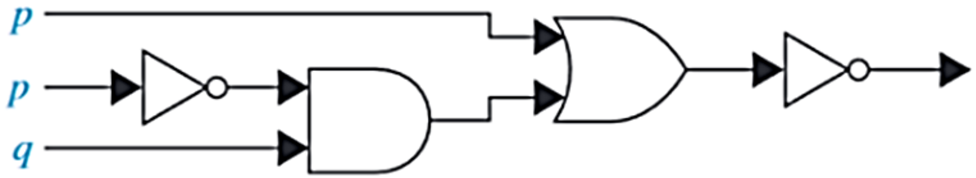
Exercise:

1. Find the output of each of these combinatorial circuits.

a)



b)



Answer: 1(a) $\neg p \vee \neg q$ **1(b)** $\neg(p \vee (\neg p \wedge q))$

2. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $(p \wedge \neg r) \vee (\neg q \wedge r)$ from input bits p, q, and r.

3. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $((\neg p \vee \neg r) \wedge \neg q) \vee (\neg p \wedge (q \vee r))$ from input bits p, q, and r.

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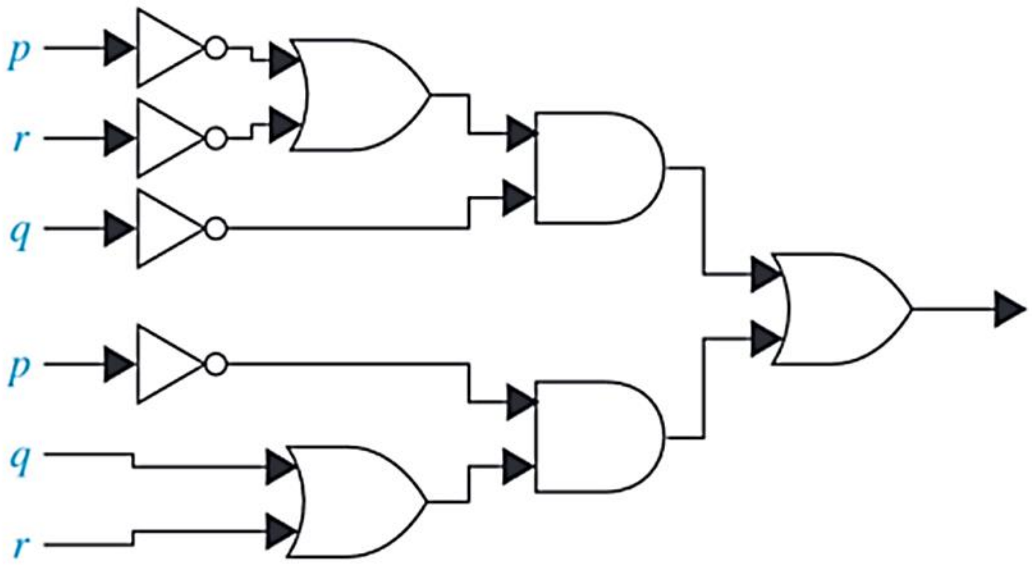
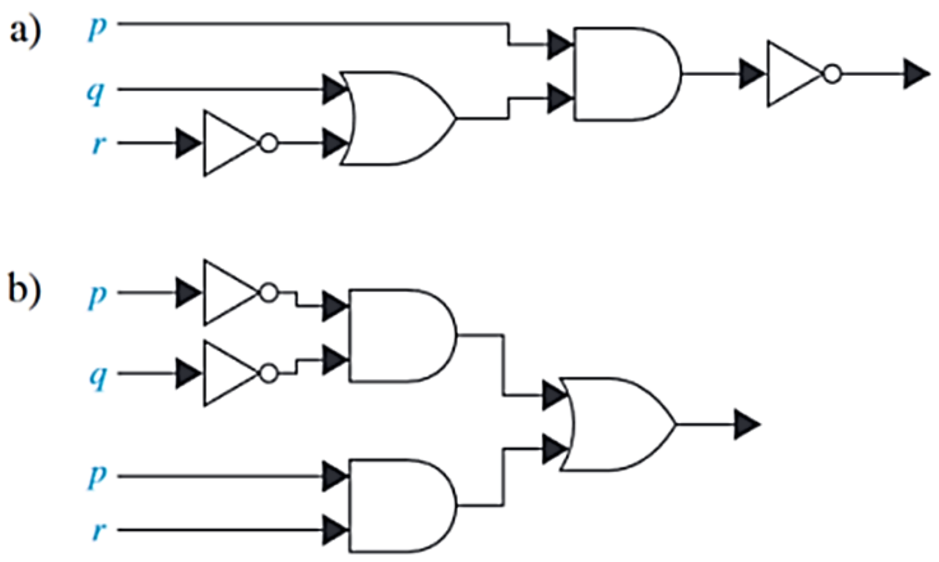
LOGIC GATES



Answer: 3

4. Find the output of each of these combinatorial circuits.



4(a) $\neg(p\wedge(q \vee \neg r))$ 4(b) $(\neg p\wedge\neg q)\vee(p\wedge r)$

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Techniques of Counting

13th WEEK

Counting Principles,
The Pigeonhole Principle

Page No: 81 - 85

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COUNTING PRINCIPLES, THE PIGEONHOLE PRINCIPLE

The Pigeonhole Principle: If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof: We prove the pigeonhole principle using a proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there is at least k + 1 objects.

Example 1: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2: In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

Example 3: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

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COUNTING PRINCIPLES, THE PIGEONHOLE PRINCIPLE

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Proof: We will use a proof by contraposition. Suppose that none of the boxes contains more than $\left\lceil \frac{N}{k} \right\rceil - 1$ objects. Then, the total number of objects is at most

$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$ where the inequality $\left\lceil \frac{N}{k} \right\rceil < \left(\frac{N}{k}\right) + 1$ has been used. This is a contradiction because there are a total of N objects.

Example: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\left\lceil \frac{N}{5} \right\rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

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COUNTING PRINCIPLES, THE PIGEONHOLE PRINCIPLE

Example: a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

Solution: a) Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$, so nine cards suffice.

b) We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worstcase, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons. Example: Find the minimum number of students in a class to be sure that three of them are born in the same month. Solution: Here the $n = 12$ months are the pigeonholes, and $k + 1 = 3$ so $k = 2$. Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

COUNTING PRINCIPLES, THE PIGEONHOLE PRINCIPLE

Question: Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

Solution:

a) If there were fewer than 9 freshmen, fewer than 9 sophomores, and fewer than 9 juniors in the class, there would be no more than 8 with each of these three class standings, for a total of at most 24 students, contradicting the fact that there are 25 students in the class.

b) If there were fewer than 3 freshmen, fewer than 19 sophomores, and fewer than 5 juniors, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

COUNTING PRINCIPLES, THE PIGEONHOLE PRINCIPLE(EXERCISE)

1. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?
2. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark. a) How many socks must he take out to be sure that he has at least two socks of the same color? b) How many socks must he take out to be sure that he has at least two black socks?
3. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them. a) How many balls must she select to be sure of having at least three balls of the same color? b) How many balls must she select to be sure of having at least three blue balls?
4. Suppose that there are 39 students in a discrete mathematics class at a small college. a) Show that the class must have at least 20 male students or at least 20 female students. b) Show that the class must have at least 13 female students or at least 27 male students.
5. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 p.m., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

Techniques of Counting

14th WEEK

Permutations, Combinations

Page No: 87 - 92

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PERMUTATIONS

Permutation: If n and r are integers with $0 \le r \le n$, then $^nP_r = \dfrac{n!}{(n-r)!}$

Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is $P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200$.

Example: Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Solution: The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour. If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths.

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COMBINATIONS

Combination: The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals $^{n}C_{r} = \frac{n!}{r!(n-r)!}$

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Solution: The answer is given by the number of 5-combinations of a set with 10 elements. The number of such combinations is $^{10}C_{5} = \frac{10!}{5!5!} = 252$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution: The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. The number of such combinations is $^{30}C_{6} = \frac{30!}{6!24!} = 593775$

Example: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. The number of ways to select the committee $^{9}C_{3} \cdot {}^{11}C_{4} = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 27720$

COMBINATIONS

Example: How many strings will form by taking 5 letters from the word "ENGINEERING"?

Solution: Given word is ENGINEERING. Number of times each letter of the word is repeated

E=3, N=3, G=2, I=2, R=1

The strings of 5 letters can be formed in the following ways:

Choosing Procedure	Combination	Permutation
(i) 3 alike and 2 distinct letters	${}^2C_1 \times {}^4C_2 = 12$	${}^2C_1 \times {}^4C_2 \times \frac{5!}{3!} = 240$
(ii) 2 alike and 3 distinct letters	${}^4C_1 \times {}^4C_3 = 16$	${}^4C_1 \times {}^4C_3 \times \frac{5!}{2!} = 960$
(iii) 2 alike pairs and 1distinct letters	${}^4C_2 \times {}^3C_1 = 18$	${}^4C_2 \times {}^3C_1 \times \frac{5!}{2!2!} = 540$
(iv) 3 alike and 2 alike letters	${}^2C_1 \times {}^3C_1 = 6$	${}^2C_1 \times {}^3C_1 \times \frac{5!}{2!3!} = 60$
(v) all distinct letters	${}^5C_5 = 1$	${}^5C_5 \times 5! = 120$
Total	**53**	**1920**

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COMBINATIONS

Example: How many strings will form by taking 5 letters from the word "**INDEPENDENT**"?

Solution: Given word is **INDEPENDENT.** Number of times each letter of the word is repeated

E=3, N=3, D=2, I=1, P=1,T=1

The strings of 5 letters can be formed in the following ways:

Choosing Procedure	Combination	Permutation
(i) 3 same and 2 different letters	${}^2C_1 \times {}^5C_2 = 20$	${}^2C_1 \times {}^5C_2 \times \frac{5!}{3!} = 400$
(ii) 2 same and 3 different letters	${}^3C_1 \times {}^5C_3 = 30$	${}^3C_1 \times {}^5C_3 \times \frac{5!}{2!} = 1800$
(iii) 2 same pairs and 1 different letters	${}^3C_2 \times {}^4C_1 = 12$	${}^4C_2 \times {}^3C_1 \times \frac{5!}{2!2!} = 360$
(iv) 2 same and 3 same letters	${}^2C_1 \times {}^2C_1 = 4$	${}^2C_1 \times {}^2C_1 \times \frac{5!}{2!3!} = 40$
(v) all different letters	${}^6C_5 = 6$	${}^6C_5 \times 5! = 720$
Total	**72**	**3320**

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PERMUTATIONS, COMBINATIONS(EXERCISE)

1. In how many ways can a set of five letters be selected from the English alphabet?
2. A store sells clothes for men. It has 3 kinds of jackets, 7 kinds of shirts, and 5 kinds of pants. Find the number of ways a person can buy: (a) one of the items; (b) one of each of the three kinds of clothes. [5.38]
3. A class has 10 male students and 8 female students. Find the number of ways the class can elect: (a) a class representative; (b) 2 class representatives, one male and one female; (c) a class president and vice president. [5.39]
4. A farmer buys 3 cows, 2 goats, and 4 hens from a man who has 6 cows, 5 goats, and 8 hens. How many ways the farmer choose his desired animals?
5. Suppose a code consists of five characters, two letters followed by three digits. Find the number of: (a) codes; (b) codes with distinct letter; (c) codes with the same letters. [5.40]
6. The English alphabet contains 21 consonants and 5 vowels. How many strings of six lowercase letters of the English alphabet contain: a) exactly one vowel? b) exactly two vowels? c) at least one vowel? d) at least two vowels?

PERMUTATIONS, COMBINATIONS(EXERCISE)

7. A class contains 10 students with 6 men and 4 women. Find the number n of ways to: (a) Select a 4-member committee from the students. (b) Select a 4-member committee with 2 men and 2 women. (c) Elect a president, vice president, and treasurer.

8. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes (a) are there in total? (b) Contain exactly two heads? (c) Contain at most three tails? (d) Contain the same number of heads and tails?

9. A club has 25 members. (a) How many ways are there to choose four members of the club to serve on an executive committee? (b) How many ways are there to choose a president, vice-president, secretary, and treasurer of the club, where no person can hold more than one office?

10. A debating team consists of 5 boys and 5 girls. Find the number of ways they can sit in a row where: (a) there are no restrictions; (b) the boys and girls are each to sit together; (c) just the girls are to sit together.

11. How many strings can be formed by taking 5 letters from the following words? (a) ASSIDUOUS (b) RIDICULOUS (c) ACCELERATOR (d) PERPETUATOR

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Graph Theory

15th WEEK

Subgraphs, Isomorphic and Homeomorphic Graphs, Paths, Connectivity

Page No: 94 - 97

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GRAPHS, SUBGRAPHS, ISOMORPHIC GRAPHS

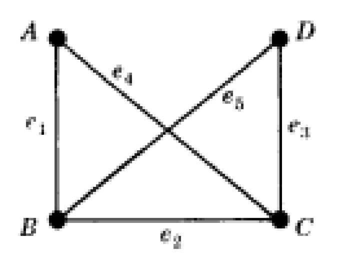

Fig. 11 : Graphs

Graphs: A graph G consists of two things:

(i) A set V = V (G) whose elements are called vertices, points, or nodes of G.

(ii) A set E = E(G) of unordered pairs of distinct vertices called edges of G.

We denote such a graph by G(V , E).

Subgraphs: Consider a graph G = G(V , E). A graph H = H (V′, E′) is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$. In particular:

(i) A subgraph H (V′, E′) of G(V , E) is called the subgraph induced by its vertices V′ if its edge set E′ contains all edges in G whose endpoints belong to vertices in H.

(ii) If v is a vertex in G, then G − v is the subgraph of G obtained by deleting v from G and deleting all edges in G which contain v.

(iii) If e is an edge in G, then G − e is the subgraph of G obtained by simply deleting the edge e from G.

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ISOMORPHIC AND HOMEOMORPHIC GRAPHS

Isomorphic Graphs: Graphs G(V , E) and G(V ∗, E∗) are said to be isomorphic if there exists a one-to-one correspondence f : V → V ∗ such that {u, v} is an edge of G if and only if {f (u), f (v)} is an edge of G∗. Normally, we do not distinguish between isomorphic graphs (even though their diagrams may "look different"). Fig.12 gives ten graphs pictured as letters. We note that A and R are isomorphic graphs. Also, F and T are isomorphic graphs, K and X are isomorphic graphs and M, S, V , and Z are isomorphic graphs.

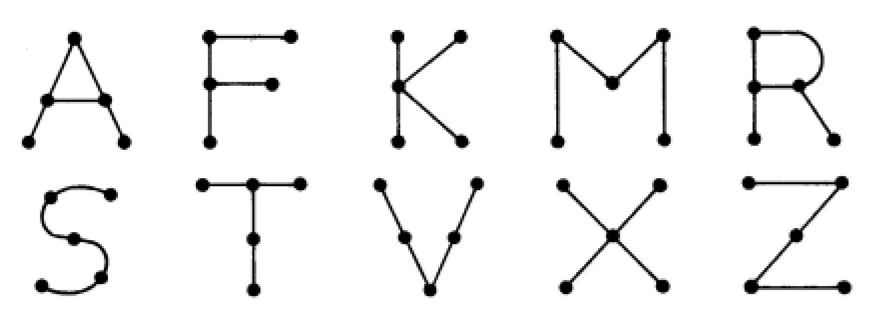

Fig. 12 : Isomorphic and Non Isomorphic Graphs

Homeomorphic Graphs: Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G∗ are said to homeomorphic if they can be obtained from the same graph or isomorphic graphs by this method. The graphs (a) and (b) in Fig 13 are not isomorphic, but they are homeomorphic since they can be obtained from the graph (c) by adding appropriate vertices.

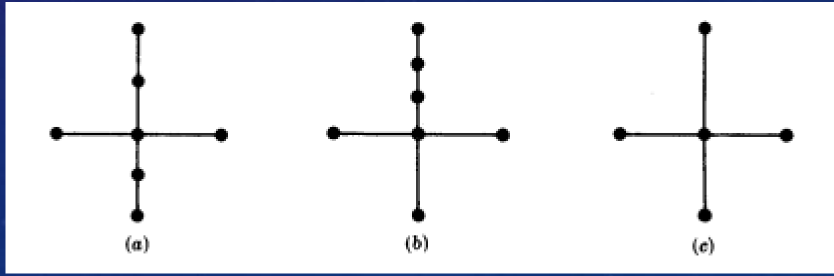

Fig. 13 : Homeomorphic and Non Homeomorphic Graphs

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PATHS, CONNECTIVITY

A *path* in a multigraph G consists of an alternating sequence of vertices and edges of the form

$$v_0, \quad e_1, \quad v_1, \quad e_2, \quad v_2, \quad \ldots, \quad e_{n-1}, \quad v_{n-1}, \quad e_n, \quad v_n$$

where each edge e_i contains the vertices v_{i-1} and v_i (which appear on the sides of e_i in the sequence). The number n of edges is called the *length* of the path. When there is no ambiguity, we denote a path by its sequence of vertices $(v_0, v_1, \ldots, v_n)$. The path is said to be *closed* if $v_0 = v_n$. Otherwise, we say the path is from v_0, to v_n or *between* v_0 and v_n, or *connects* v_0 to v_n.

A *simple path* is a path in which all vertices are distinct. (A path in which all edges are distinct will be called a *trail*.) A *cycle* is a closed path of length 3 or more in which all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a *k-cycle*.

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PATHS, CONNECTIVITY

EXAMPLE 8.1 Consider the graph G in Fig. 8-8(a). Consider the following sequences:

$$\alpha = (P_4, P_1, P_2, P_5, P_1, P_2, P_3, P_6), \quad \beta = (P_4, P_1, P_5, P_2, P_6),$$
$$\gamma = (P_4, P_1, P_5, P_2, P_3, P_5, P_6), \quad \delta = (P_4, P_1, P_5, P_3, P_6).$$

The sequence α is a path from P_4 to P_6; but it is not a trail since the edge $\{P_1, P_2\}$ is used twice. The sequence β is not a path since there is no edge $\{P_2, P_6\}$. The sequence γ is a trail since no edge is used twice; but it is not a simple path since the vertex P_5 is used twice. The sequence δ is a simple path from P_4 to P_6; but it is not the shortest path (with respect to length) from P_4 to P_6. The shortest path from P_4 to P_6 is the simple path (P_4, P_5, P_6) which has length 2.

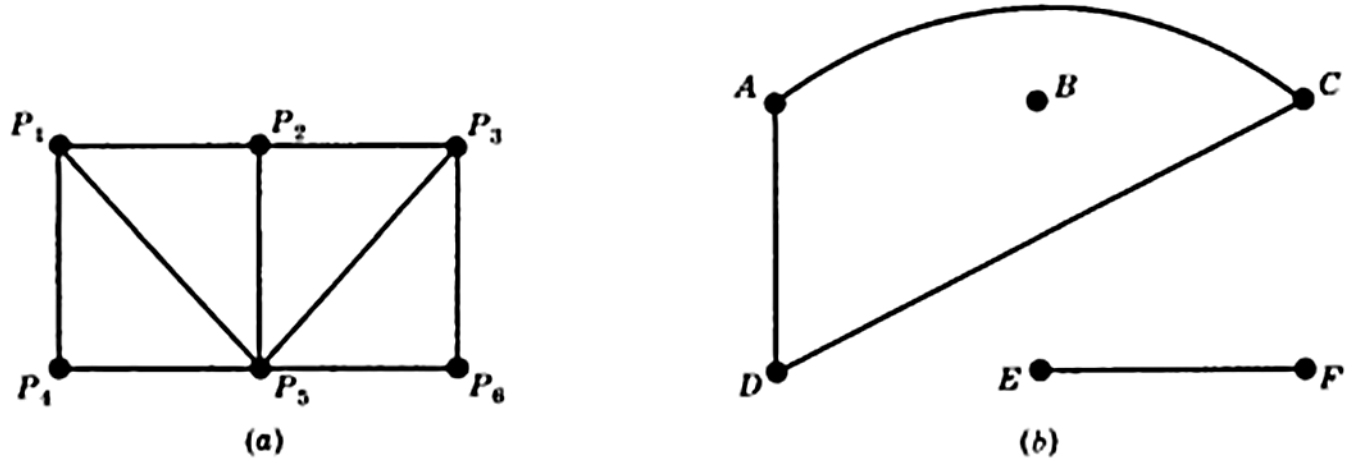

Fig. 8-8

Graph Theory

16th WEEK

Labeled and Weighted Graphs, Complete, Regular, and Bipartite Graphs

Page No: 99 - 102

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LABELED AND WEIGHTED GRAPHS

A graph G is called a *labeled graph* if its edges and/or vertices are assigned data of one kind or another. In particular, G is called a *weighted graph* if each edge e of G is assigned a nonnegative number $w(e)$ called *the weight* or *length* of v. Figure 8-12 shows a weighted graph where the weight of each edge is given in the obvious way. The *weight* (or *length*) of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path. One important problem in graph theory is to find a *shortest path*, that is, a path of minimum weight (length), between any two given vertices. The length of a shortest path between P and Q in Fig. 8-12 is 14; one such path is

$$(P, A_1, A_2, A_5, A_3, A_6, Q)$$

The reader can try to find another shortest path.

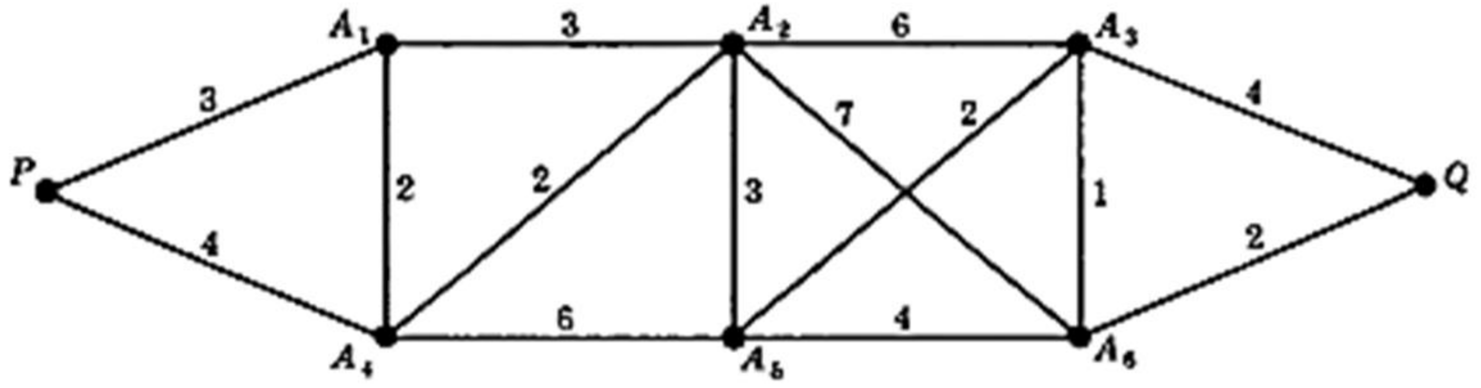

Fig. 8-12

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COMPLETE GRAPHS

Complete Graphs: A graph G is said to be complete if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by K_n . Fig 8-13 shows the graphs K_1 through K_6 .

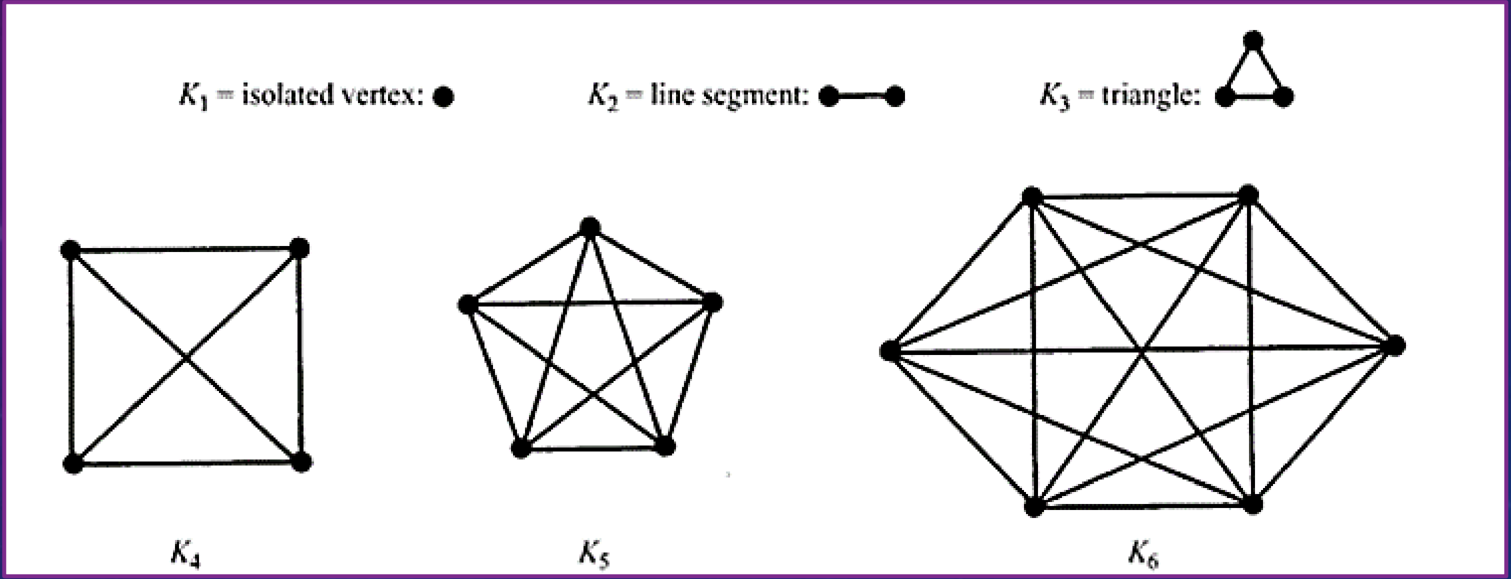

Fig. 8-13

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REGULAR GRAPHS

Regular Graphs: A graph G is regular of degree k or k-regular if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree. The connected regular graphs of degrees 0, 1, or 2 are easily described. The connected 0-regular graph is the trivial graph with one vertex and no edges. The connected 1-regular graph is the graph with two vertices and one edge connecting them. The connected 2-regular graph with n vertices is the graph which consists of a single n-cycle. See Fig. 8-14.

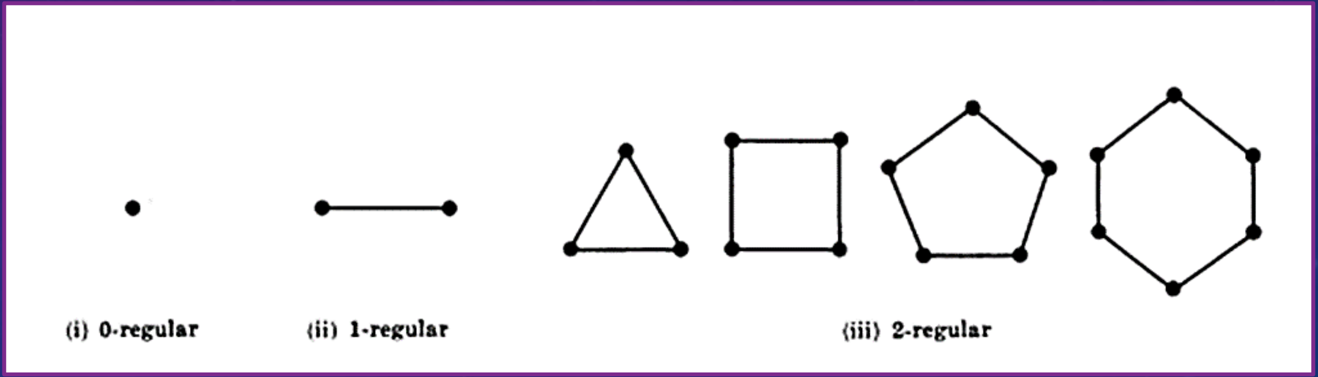

Fig. 8-14

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BIPARTITE GRAPHS

Bipartite Graphs: A graph G is said to be bipartite if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N. By a complete bipartite graph, we mean that each vertex of M is connected to each vertex of N; this graph is denoted by $K_{m,n}$ where m is the number of vertices in M and n is the number of vertices in N, and, for standardization, we will assume m ≤ n. Figure 8-16 shows the graphs $K_{2,3}$, $K_{3,3}$, and $K_{2,4}$, Clearly the graph $K_{m,n}$ has mn edges.

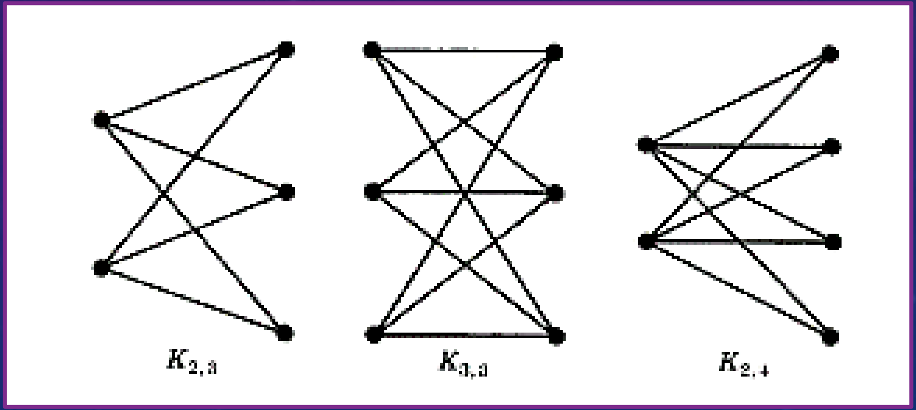

Fig. 8-16

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Graph Theory

17th WEEK

Tree Graphs, Planar Graphs,

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TREE GRAPHS

A graph T is called a tree if T is connected and T has no cycles. Examples of trees are shown in Fig. 8-17. A forest G is a graph with no cycles; hence the connected components of a forest G are trees. A graph without cycles is said to be cycle-free. The tree consisting of a single vertex with no edges is called the degenerate tree.

Consider a tree T . Clearly, there is only one simple path between two vertices of T ; otherwise, the two paths would form a cycle. Also:

(a) Suppose there is no edge {u, v} in T and we add the edge e = {u, v} to T . Then the simple path from u to v in T and e will form a cycle; hence T is no longer a tree.

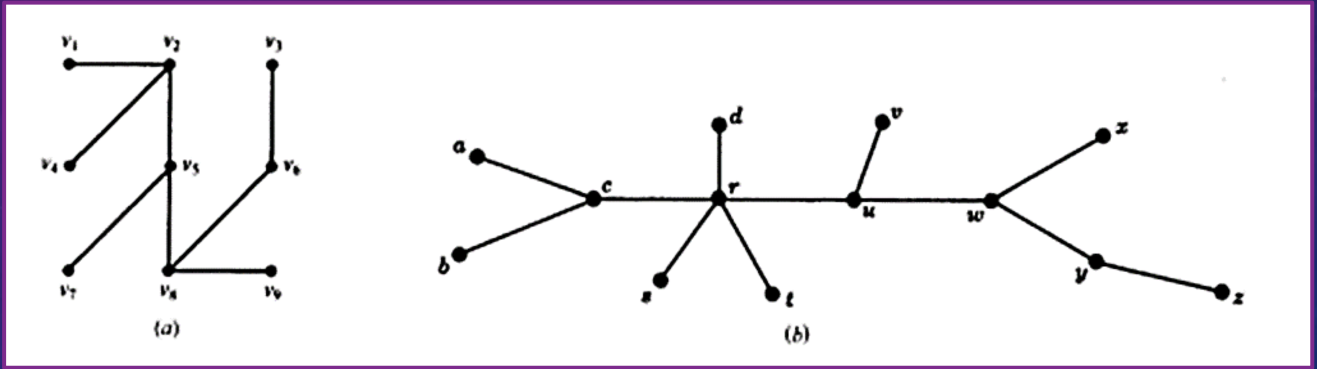

Fig. 8-17

(b) On the other hand, suppose there is an edge e = {u, v} in T , and we delete e from T . Then T is no longer connected (since there cannot be a path from u to v); hence T is no longer a tree.

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PLANAR GRAPHS

A graph or multigraph which can be drawn in the plane so that its edges do not cross is said to be planar. Although the complete graph with four vertices K_4 is usually pictured with crossing edges as in Fig. 8-21(a), it can also be drawn with noncrossing edges as in Fig. 8-21(b); hence K_4 is planar. Tree graphs form an important class of planar graphs. This section introduces our reader to these important graphs.

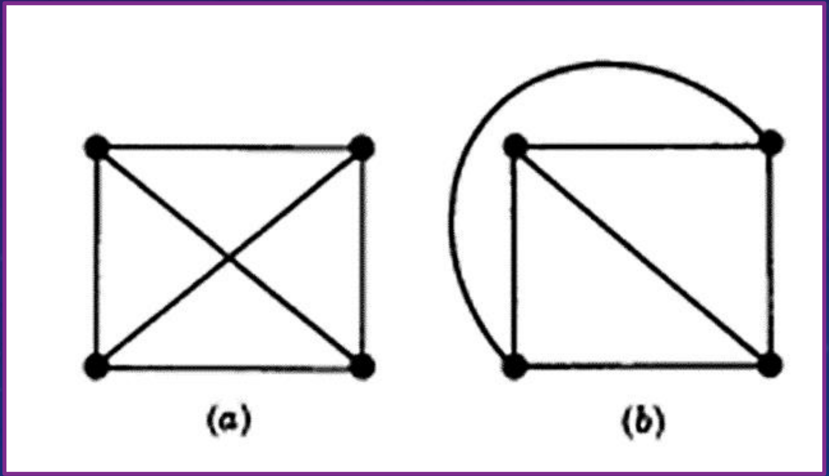

Fig. 8-21

Thanks For Being With Me

THE END

Wish You A Happy Journey In Learning Mathematics.

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